Theorem: π^2 (and therefore π) is irrational (As if π were rational π^2 would be too). Because rational numbers are exactly those whose decimals repeat (proven in Level 7.1), π's decimals never repeat.

Levels recommended for proof: 3-5

Proof:

Start by assuming $\pi^2 = \frac{a}{b}$, we aim to derive a contradiction. Define the sequence of integrals I_n as follows:

- $I_0 = 2$
- $I_1 = 4b$
- $I_n = \frac{b^n}{n!}\int_0^\pi x^n(\pi - x)^n \sin(x)\,dx \ \ (n \geq 2)$

We can bound I_n as follows: It is non-negative since everything in the integral is non-negative and so is the coefficient, and $x^n(\pi - x)^n$ is bounded above by $\pi^n\pi^n = \pi^{2n}$ whenever x is between 0 and π, and therefore the integral can be bounded above by $\int_0^\pi \pi^{2n}\sin(x)\,dx = \pi^{2n}\int_0^\pi \sin(x)\,dx = 2\pi^{2n}$. Therefore we can bound I_n by $\frac{2(\pi^2 b)^n}{n!}$. Note that when, for example, $n > 2\pi^2 b$, the terms $\frac{2(\pi^2 b)^n}{n!}$ will be at most half of the previous term (since the numerator will be multiplied by $\pi^2 b$ and the denominator by more than twice that), therefore these terms eventually tend to 0. Since I_n is bounded above by these terms and never negative, I_n also tends to 0.

The idea of the proof will be to show that if $\pi^2 = \frac{a}{b}$ then I_n is an integer for all n, which is a problem since I_n is always strictly between 0 and something that tends to 0 (and therefore is less than 1), so I_n can never be an integer and this will be our contradiction.

Now we will do a classic A level further maths problem where we show from algebra that for n at least 2, $I_n = 2b(2n-1)I_{n-1} - abI_{n-2}$, which would mean that I_n is always an integer (since a, b, n, and I_0, I_1 are all integers) so we will be done.

It turns out that $I_n = \frac{b^n}{n!}\int_0^\pi x^n(\pi - x)^n \sin(x)\,dx$ for n=0 and n=1. The proof is that for n=0, the integral simplifies to $\frac{b^0}{0!}\int_0^\pi x^0(\pi - x)^0 \sin(x)\,dx = \int_0^\pi \sin(x)\,dx = 2$ (it's kind of cool that the area under one of the waves of the sine function is exactly 2 isn't it), and for n=1, the integral simplifies to $b\int_0^\pi x(\pi - x)\sin(x)\,dx = b\pi\int_0^\pi x\sin(x)\,dx - b\int_0^\pi x^2\sin(x)\,dx$. Integration by parts on the first integral setting $u = x, \frac{dv}{dx} = \sin(x)$ so $\frac{du}{dx} = 1, v = -\cos(x)$ gives that it is exactly equal to $-[x\cos(x)]_0^\pi - \int_0^\pi -\cos(x)\,dx = -(-\pi - 0) - 0 = \pi$. Also, $\int_0^\pi x^2\sin(x)\,dx$ can be integrated by parts if we set $u = x^2, dv = \sin(x)\,dx$ so $du = 2xdx, v = -\cos(x)$, meaning that the integral is then equal to $-[x^2\cos(x)]_0^\pi - \int_0^\pi -2x\cos(x)\,dx = -(-\pi^2 - 0) + \int_0^\pi 2x\cos(x)\,dx = \pi^2 + \int_0^\pi 2x\cos(x)\,dx$. Integrating the second integral by parts, setting $u = 2x, \frac{dv}{dx} = \cos(x)$ so $\frac{du}{dx} = 2, v = \sin(x)$, we get that $\int_0^\pi 2x\cos(x)\,dx = [2x\sin(x)]_0^\pi - \int_0^\pi 2\sin(x)\,dx = (0-0) - 4 = -4$, therefore the original integral $\int_0^\pi x^2\sin(x)\,dx$ is equal to $\pi^2 - 4$. Now $I_1 = b\pi(\pi) - b(\pi^2 - 4) = 4b$, as required.

Now the final step is to prove the recurrence relation $I_n = 2b(2n-1)I_{n-1} - abI_{n-2}$ for n at least 2.

Consider $\int_0^\pi x^n(\pi-x)^n\sin(x)\,dx$. We will do this by parts. Set $u=x^n(\pi-x)^n$, $\frac{dv}{dx}=\sin(x)$. By the product and chain rules, $\frac{du}{dx}=nx^{n-1}(\pi-x)^n-nx^n(\pi-x)^{-1}=x^{n-1}(\pi-x)^{n-1}[n(\pi-x)-nx]=nx^{n-1}(\pi-x)^{n-1}(\pi-2x)$. By the integration by parts formula $\int_0^\pi u\frac{dv}{dx}dx=[uv]_0^\pi-\int_0^\pi v\frac{du}{dx}dx$, but u vanishes (is 0) at 0 and π and therefore uv does too. Therefore $\int_0^\pi u\frac{dv}{dx}dx=-\int_0^\pi v\frac{du}{dx}dx$. Therefore, since $v=-\cos(x)$, $\int_0^\pi x^n(\pi-x)^n\sin(x)\,dx=n\int_0^\pi x^{n-1}(\pi-x)^{n-1}(\pi-2x)\cos(x)\,dx$. Now we want to integrate this by parts again so work out the derivative of $x^{n-1}(\pi-x)^{n-1}(\pi-2x)$. By the product rule on the last term, this is equal to $-2x^{n-1}(\pi-x)^{n-1}+(\pi-2x)\frac{d}{dx}[x^{n-1}(\pi-x)^{n-1}]$, but this last derivative is just the n-1 version of something whose derivative we already know, so $\frac{d}{dx}[x^{n-1}(\pi-x)^{n-1}]=(n-1)x^{n-2}(\pi-x)^{n-2}(\pi-2x)$. Therefore the original derivative we wanted, ie the derivative of $x^{n-1}(\pi-x)^{n-1}(\pi-2x)$, is $-2x^{n-1}(\pi-x)^{n-1}+(\pi-2x)^2(n-1)x^{n-2}(\pi-x)^{n-2}$. Therefore for $\int_0^\pi x^{n-1}(\pi-x)^{n-1}(\pi-2x)\cos(x)\,dx$ we set $u=x^{n-1}(\pi-x)^{n-1}$, $\frac{dv}{dx}=\cos(x)$, so now we have $v=\sin(x)$, $\frac{du}{dx}=-2x^{n-1}(\pi-x)^{n-1}+(\pi-2x)^2(n-1)x^{n-2}(\pi-x)^{n-2}$. This time v vanishes at 0 and π so we reduce to $-\int_0^\pi v\frac{du}{dx}dx$

$=\int_0^\pi[2x^{n-1}(\pi-x)^{n-1}-(\pi-2x)^2(n-1)x^{n-2}(\pi-x)^{n-2}]\sin(x)dx$. By some algebra, this is $\int_0^\pi[2x^{n-1}(\pi-x)^{n-1}-\left(\pi^2-4x(\pi-x)\right)(n-1)x^{n-2}(\pi-x)^{n-2}]\sin(x)dx$. This is therefore equal to $\int_0^\pi[(2+4(n-1))x^{n-1}(\pi-x)^{n-1}-(\pi^2)(n-1)x^{n-2}(\pi-x)^{n-2}]\sin(x)dx$.

Ok so putting all this together

$$I_n=\frac{b^n}{n!}\int_0^\pi x^n(\pi-x)^n\sin(x)\,dx=\frac{b^n}{(n)!}n\int_0^\pi x^{n-1}(\pi-x)^{n-1}(\pi-2x)\cos(x)\,dx$$

$$=\frac{b^n}{(n-1)!}\left[\int_0^\pi\left[\left(2+4(n-1)\right)x^{n-1}(\pi-x)^{n-1}-(\pi^2)(n-1)x^{n-2}(\pi-x)^{n-2}\right]\sin(x)\,dx\right]$$

$$=b\frac{b^{n-1}}{(n-1)!}\int_0^\pi\left[\left(2+4(n-1)\right)x^{n-1}(\pi-x)^{n-1}\right]\sin(x)\,dx-\frac{b^2\pi^2}{n-1}\frac{b^{n-2}}{(n-2)!}\left[\int_0^\pi[(n-1)x^{n-2}(\pi-x)^{n-2}]\sin(x)\,dx\right]$$

$$=bI_{n-1}\left(2+4(n-1)\right)-b^2\pi^2\frac{b^{n-2}}{(n-2)!}\left[\int_0^\pi[x^{n-2}(\pi-x)^{n-2}]\sin(x)\,dx\right]$$

$=2b(2n-1)I_{n-1}-abI_{n-2}$ where we have used the fact that $\pi^2=\frac{a}{b}$. This completes the proof of the formula so we are done.